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# Mathematics: analysis and approaches

## Higher level

### Paper 3

9 May 2023

Zone A afternoon | Zone B morning | Zone C afternoon

1 hour

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

**This question asks you to examine the number and nature of intersection points of the graph of  $y = \log_a x$  where  $a \in \mathbb{R}^+$ ,  $a \neq 1$  and the line  $y = x$  for particular sets of values of  $a$ .**

In this question you may either use the change of logarithm base formula  $\log_a x = \frac{\ln x}{\ln a}$  or a graphic display calculator “logarithm to any base feature”.

The function  $f$  is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+, a \neq 1.$$

(a) Consider the cases  $a = 2$  and  $a = 10$ . On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero  $x$ -axis intercepts.

[4]

**(This question continues on the following page)**

**(Question 1 continued)**

In parts (b) and (c), consider the case where  $a = e$ . Note that  $\ln x \equiv \log_e x$ .

- (b) Use calculus to find the minimum value of the expression  $x - \ln x$ , justifying that this value is a minimum. [5]
- (c) Hence deduce that  $x > \ln x$ . [1]
- (d) There exist values of  $a$  for which the graph of  $y = \log_a x$  and the line  $y = x$  do have intersection points. The following table gives three intervals for the value of  $a$ .

Interval	Number of intersection points
$0 < a < 1$	$p$
$1 < a < 1.4$	$q$
$1.5 < a < 2$	$r$

By investigating the graph of  $y = \log_a x$  for different values of  $a$ , write down the values of  $p$ ,  $q$  and  $r$ . [4]

In parts (e) and (f), consider  $a \in \mathbb{R}^+$ ,  $a \neq 1$ .

For  $1.4 \leq a \leq 1.5$ , a value of  $a$  exists such that the line  $y = x$  is a tangent to the graph of  $y = \log_a x$  at a point P.

- (e) Find the exact coordinates of P and the exact value of  $a$ . [8]
- (f) Write down the exact set of values for  $a$  such that the graphs of  $y = \log_a x$  and  $y = x$  have
  - (i) two intersection points; [1]
  - (ii) no intersection points. [1]

**Turn over**

2. [Maximum mark: 31]

**This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.**

Consider the function  $L(x) = mx + c$  for  $x \in \mathbb{R}$  where  $m, c \in \mathbb{R}$  and  $m, c \neq 0$ .

Let  $r \in \mathbb{R}$  be the root of  $L(x) = 0$ .

If  $m, r$  and  $c$ , in that order, are in arithmetic sequence then  $L(x)$  is said to be an AS-linear function.

(a) Show that  $L(x) = 2x - 1$  is an AS-linear function. [2]

Consider  $L(x) = mx + c$ .

(b) (i) Show that  $r = -\frac{c}{m}$ . [1]

(ii) Given that  $L(x)$  is an AS-linear function, show that  $L(x) = mx - \frac{m^2}{m+2}$ . [4]

(iii) State any further restrictions on the value of  $m$ . [1]

There are only three **integer** sets of values of  $m, r$  and  $c$ , that form an AS-linear function. One of these is  $L(x) = -x - 1$ .

(c) Use part (b) to determine the other two AS-linear functions with integer values of  $m, r$  and  $c$ . [3]

Consider the function  $Q(x) = ax^2 + bx + c$  for  $x \in \mathbb{R}$  where  $a \in \mathbb{R}, a \neq 0$  and  $b, c \in \mathbb{R}$ .

Let  $r_1, r_2 \in \mathbb{R}$  be the roots of  $Q(x) = 0$ .

(d) Write down an expression for

(i) the sum of roots,  $r_1 + r_2$ , in terms of  $a$  and  $b$ . [1]

(ii) the product of roots,  $r_1 r_2$ , in terms of  $a$  and  $c$ . [1]

**(This question continues on the following page)**

**(Question 2 continued)**

If  $a, r_1, b, r_2$  and  $c$ , in that order, are in arithmetic sequence, then  $Q(x)$  is said to be an AS-quadratic function.

- (e) Given that  $Q(x)$  is an AS-quadratic function,
- (i) write down an expression for  $r_2 - r_1$  in terms of  $a$  and  $b$ ; [1]
  - (ii) use your answers to parts (d)(i) and (e)(i) to show that  $r_1 = \frac{a^2 - ab - b}{2a}$ ; [2]
  - (iii) use the result from part (e)(ii) to show that  $b = 0$  or  $a = -\frac{1}{2}$ . [3]

Consider the case where  $b = 0$ .

- (f) Determine the two AS-quadratic functions that satisfy this condition. [5]

Now consider the case where  $a = -\frac{1}{2}$ .

- (g) (i) Find an expression for  $r_1$  in terms of  $b$ . [2]
- (ii) Hence or otherwise, determine the exact values of  $b$  and  $c$  such that AS-quadratic functions are formed.
- Give your answers in the form  $\frac{-p \pm q\sqrt{s}}{2}$  where  $p, q, s \in \mathbb{Z}^+$ . [5]

**References:**